GROUP ACTION PRESERVING HAAGERUP PROPERTY OF C^*-ALGEBRAS

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Abstract. In this short note, from the viewpoint of C^*-dynamical system, we define a weak version of Haagerup property for group action on C^*-algebra. It is proved that this kind of group action preserves Haagerup property of C^*-algebras in the sense of Z. Dong, i.e., the reduced crossed product C^*-algebra $A \rtimes_{\alpha,\mathrm{r}} \Gamma$ has Haagerup property with respect to the induced faithful tracial state $\widetilde{\tau}$ if A has Haagerup property with respect to τ.

1. Introduction and Preliminaries

It is well-known that a discrete group Γ is amenable if and only if its reduced group C^*-alegbra $C^*_{\mathrm{r}}(\Gamma)$ is nuclear [1], which reflects that amenability in geometric group theory corresponds to nuclearity in operator algebra. Similar correspondence is expected for Haagerup property [2], which is certain weak version of amenability (e.g., equivalent to Gromov's a-T-menability [3]). To this end, motivated by the definition of Haagerup property of von Neumann algebra [4], Z. Dong recently defined Haagerup property for C^*-algebra as follows.

Definition 1.1 (Dong, [5])**.** Let A be a unital C^*-algebra, τ a faithful tracial state on A. The pair (A, τ) is said to have *Haagerup property* if there is a net $(\phi_i)_{i\in I}$ of u.c.p. maps from A to itself satisfying the following conditions:

(1) Each ϕ_i decreases τ; i.e., for any $a \in A^+$, we have $\tau(\phi_i(a)) \leq \tau(a)$;

(2) For any $a \in A$, $\|\phi_i(a) - a\|_\tau$ converges to 0 as i tends to infinity;

(3) Each ϕ_i is L^2-compact; i.e., from the first condition, ϕ_i extends to a bounded operator on its GNS-space $L^2(A, \tau)$, which is compact.

With this definition, Dong successfully set up the following correspondence that is similar to the case of amenability and nuclearity: for a discrete group Γ, it has Haagerup property if and only if its reduced group C^*-algebra $C^*_{\mathrm{r}}(\Gamma)$ has Haagerup property with respect to the canonical tracial state [5]. Moreover, Dong also explored the behavior of Haagerup property of C^*-algebra in C^*-dynamical system and achieved the following theorem:

Theorem 1.2 (Dong, [5])**.** *Suppose that A is a unital separable C^*-algebra with a faithful tracial state τ and a τ-preserving action α of a countable discrete group*

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Γ. *If* Γ *is amenable and* (A,τ) *has Haagerup property, then* $(A \rtimes_{\alpha,r} \Gamma, \tau')$ *also has Haagerup property, where* τ' *is the induced trace of* τ *on* $A \rtimes_{\alpha,r} \Gamma$.

The proof of Theorem 1.2 relies on an approximation technique from Lemma 4.3.3 of [6], where Følner sets play a crucial role, therefore group Γ is supposed to be amenable. Meanwhile, we also notice that [6] gives the concept of *amenable action* of discrete group on C^*-algebra (Definition 4.3.1) and proves that: if $\alpha : \Gamma \curvearrowright A$ is amenable, then A is nuclear if and only if $A \rtimes_{\alpha,\mathrm{r}} \Gamma$ is nuclear (Theorem 4.3.4). Inspired by this theorem, we want to define certain Haagerup property for discrete group action on C^*-algebra such that: (1)this kind of group action will preserve Haagerup property of C^*-algebras; (2)action of group with Haagerup property naturally possesses this property.

Actually, there is some work to track. In [7], Z. Dong and Z.-J. Ruan defined the following Haagerup property of discrete group action on C^*-algebra.

Definition 1.3 (Dong-Ruan, [7])**.** We say that the action $\alpha : \Gamma \curvearrowright A$ has *Haagerup property* if there exists a sequence of completely positive multipliers $\{h_n\}$ in $C_0(\Gamma, A)$ such that $h_n \to 1$ pointwisely on Γ.

Evidently, this definition is motivated by that of group's Haagerup property, with $h_n \in c_0(\Gamma)$ replaced by $h_n \in C_0(\Gamma, A)$. Analogous to Theorem 2.6 of [5], Dong and Ruan proved that the action $\alpha : \Gamma \curvearrowright A$ has Haagerup property if and only if the reduced crossed product $A \rtimes_{\alpha,\mathrm{r}} \Gamma$ has the Hilbert A-module Haagerup property (Theorem 3.6, [7]). Though this is a delicate generalization of discrete group's Haagerup property, C^*-algebra A in Definition 1.3 only functions as scalar so that little attention is paid to its role in C^*-dynamical system. In order to relate Haagerup property of $A \rtimes_{\alpha,\mathrm{r}} \Gamma$ with that of A, we need to modify Definition 1.3 such that information about Haagerup property of A must be involved.

$\widetilde{\tau} = \tau \circ \mathcal{E}$ is the induced faithful tracial state of τ on $A \rtimes_{\alpha,\mathrm{r}} \Gamma$, where $\mathcal{E} : A \rtimes_{\alpha,\mathrm{r}} \Gamma \to A$ is the canonical faithful conditional expectation. Let $h : \Gamma \to \mathcal{Z}(A)$ be a bounded positive definite map with respect to $\alpha : \Gamma \curvearrowright A$. We say that h is vanishing at infinity with respect to a faithful tracial state τ on A, denoted as $h \in C_{0,\tau}(\Gamma, A)$, if for arbitrary $\varepsilon > 0$, there exists a finite subset $F \subseteq \Gamma$ such that $\|h_s\|_{2,\tau} < \varepsilon$ for all $s \notin F$, where $\|\cdot\|_{2,\tau}$ is the L^2-norm on the GNS-space $L^2(A,\tau)$ induced by τ.

Definition 1.4. We say that the action $\alpha : \Gamma \curvearrowright A$ has *Haagerup property with respect to a faithful tracial state* τ on A if there exists a sequence of bounded positive definite maps $\{h_n : \Gamma \to \mathcal{Z}(A)\}$ in $C_{0,\tau}(\Gamma, A)$ such that $h_n \to 1$ pointwisely with respect to τ (i.e. $\|h_n(s) - 1\|_{2,\tau} \to 0$ as $n \to \infty$ for any $s \in \Gamma$).

With this innovated definition, we are ready to present the main theorem of this paper.

Theorem 1.5. *Let* Γ *be a countable discrete group and* α *be an action of* Γ *on a unital* C^**-algebra* A *such that* α *has Haagerup property with respect to a faithful tracial state* τ *on* A*, then the reduced crossed product* $A \rtimes_{\alpha,r} \Gamma$ *has Haagerup property with respect to the induced faithful tracial state* $\widetilde{\tau}$ *if* A *has Haagerup property with respect to* τ.

Remark 1.6. Suppose that discrete group Γ has Haagerup property, i.e., there exists a sequence of positive definite functions $\{\varphi_i\}$ on Γ with $\varphi_i(e) = 1$, such that each φ_i vanishes at infinity and $\varphi_i \to 1$ pointwisely, and $\alpha : \Gamma \curvearrowright A$ is the action of Γ on C^*-algebra A. If we set $h_n : \Gamma \to A$ as $h_n = \varphi_n 1$, then it is trivial that, equipped with $\{h_n\}$, $\alpha : \Gamma \curvearrowright A$ would have Haagerup property with respect to any faithful tracial state τ on A, that is exactly what we expected. Consequently, Theorem 1.2 immediately becomes a corollary of our theorem.

2. Proof of the Main Theorem

Suppose that A has Haagerup property with respect to a faithful tracial state τ, then there exists a sequence of unital completely positive maps $\{\varphi_n\}$ from A into itself such that

(1) $\tau \circ \varphi_n \leq \tau$ and φ_n is $L_{2,\tau}$-compact for every n;
(2) For every $a \in A$, $\|\varphi_n(a) - a\|_{2,\tau} \to 0$ as $n \to \infty$.

We will show that $\{\Phi_n\}$ defined by $\Phi_n(x) = \sum_{t\in\Gamma} \lambda_t \varphi_n(a_t) h_n(t)$, where $x = \sum_{t\in\Gamma} \lambda_t a_t \in C_c(\Gamma, A)$, will fulfill the requirement for $A \rtimes_{\alpha,\mathrm{r}} \Gamma$ to have Haagerup property with respect to $\widetilde{\tau}$. Please note that each φ_n is completely positive, hence $\{\Phi_n\}$ is also completely positive on $A \rtimes_{\alpha,\mathrm{r}} \Gamma$ according to the proof of Theorem 3.2, [7]. Let $\widetilde{\Phi}_n$ denote the map on $L^2(A \rtimes_{\alpha,\mathrm{r}} \Gamma, \widetilde{\tau})$ induced by Φ_n. We still have $\|\widetilde{\Phi}_n\|_{2,\widetilde{\tau}} \leq \|\Phi\|$ (corresponding to Proposition 3.3, [7], with a minor modification that $\tau \circ \mathcal{E}(\Phi^*(x)\Phi(x)) \leq \|\Phi\|^2 \tau \circ \mathcal{E}(x^* x)$).

Lemma 2.1. *Suppose that A is a unital C^*-algebra with faithful tracial state τ, $\varphi : A \to A$ is unital and $L_{2,\tau}$-compact, $h : \Gamma \to \mathcal{Z}(A)$ is a bounded positive definite map with respect to group action $\alpha : \Gamma \curvearrowright A$. Let $\Phi : A \rtimes_{\alpha,r} \Gamma \to A \rtimes_{\alpha,r} \Gamma$ be defined by $\Phi(x) = \sum_{t\in\Gamma} \lambda_t \varphi(a_t) h(t)$, where $x = \sum_{t\in\Gamma} \lambda_t a_t \in C_c(\Gamma, A)$. Then h is vanishing at infinity with respect to τ if and only if the induced map $\widetilde{\Phi}$ is compact on $L^2(A \rtimes_{\alpha,r} \Gamma, \widetilde{\tau})$.*

Proof. "⇒" Assume that h is vanishing at infinity with respect to τ. For any integer $k > 0$, we can find finite subset $F_k \subseteq \Gamma$ such that $\|h(t)\|_{2,\tau} < 1/k$ for any $t \notin F_k$. Given that $\varphi : A \to A$ is $L_{2,\tau}$-compact, we can find a sequence of finite-rank maps $\{\varphi_k : A \to A\}_{k\in\mathbb{N}}$ such that $\|\widetilde{\varphi} - \widetilde{\varphi}_k\|_{2,\tau} < 1/k$. Let $T_k(x) = \sum_{t\in F_k} \lambda_t \varphi_k(a_t) h(t)$, where $x = \sum_{t\in\Gamma} \lambda_t a_t$ ($\widetilde{T}_k$ is indeed finite-rank because so is φ_k). So, for any $x = \sum_{t\in\Gamma} \lambda_t a_t \in C_c(\Gamma, A)$, $\Phi(x) - T_k(x) = \sum_{t\in F_k} \lambda_t (\varphi - \varphi_k)(a_t) h(t) + \sum_{t\notin F_k} \lambda_t \varphi(a_t) h(t)$.

Let $E_1 = \sum_{t\in F_k} \lambda_t (\varphi - \varphi_k)(a_t) h(t)$, $E_2 = \sum_{t\notin F_k} \lambda_t \varphi(a_t) h(t)$. So $\|\widetilde{\Phi}(x) - \widetilde{T}_k(x)\|_{2,\widetilde{\tau}}^2 \leq 2(\|E_1\|_{2,\widetilde{\tau}}^2 + \|E_2\|_{2,\widetilde{\tau}}^2)$.

$$\begin{aligned}
\|E_1\|_{2,\widetilde{\tau}}^2 =&\tau(\sum_{t\in F_k} h^*(t)(\varphi-\varphi_k)^*(a_t)(\varphi-\varphi_k)(a_t)h(t))\\
=&\sum_{t\in F_k}\tau(h^*(t)(\varphi-\varphi_k)^*(a_t)(\varphi-\varphi_k)(a_t)h(t))\\
\leq&\sum_{t\in F_k}\tau((\varphi-\varphi_k)^*(a_t)(\varphi-\varphi_k)(a_t))\tau(h^*(t)h(t))\\
=&\sum_{t\in F_k}\|(\widetilde{\varphi}-\widetilde{\varphi}_k)(a_t)\|_{2,\tau}^2\|h(t)\|_{2,\tau}^2\\
<&\sum_{t\in F_k}\frac{1}{k^2}\|a_t\|_{2,\tau}^2 M \qquad (\text{where } M=\max_{t\in\Gamma}\|h(t)\|_{2,\tau}^2)\\
=&\frac{M}{k^2}\sum_{t\in F_k}\tau(a_t^*a_t)\leq\frac{M}{k^2}\sum_{t\in\Gamma}\tau(a_t^*a_t)=\frac{M}{k^2}\|x\|_{2,\widetilde{\tau}}^2,
\end{aligned}$$

$$\begin{aligned}
\|E_2\|_{2,\widetilde{\tau}}^2 =&\tau(\sum_{t\notin F_k} h^*(t)\varphi^*(a_t)\varphi(a_t)h(t))\\
=&\sum_{t\notin F_k}\tau(h^*(t)\varphi^*(a_t)\varphi(a_t)h(t))\\
\leq&\sum_{t\notin F_k}\tau(\varphi^*(a_t)\varphi(a_t))\tau(h^*(t)h(t))\\
=&\sum_{t\notin F_k}\|\widetilde{\varphi}(a_t)\|_{2,\tau}^2\|h(t)\|_{2,\tau}^2\\
\leq&\sum_{t\notin F_k}\|\widetilde{\varphi}\|_{2,\tau}^2\|a_t\|_{2,\tau}\frac{1}{k^2}\\
\leq&\frac{\|\widetilde{\varphi}\|_{2,\tau}^2}{k^2}\|x\|_{2,\widetilde{\tau}}^2.
\end{aligned}$$

Thus, $\|\widetilde{\Phi}(x)-\widetilde{T}_k(x)\|_{2,\widetilde{\tau}}^2 \leq \frac{\sqrt{2(M+\|\widetilde{\varphi}\|_{2,\tau}^2)}}{k}\|x\|_{2,\widetilde{\tau}}$. This shows that $\widetilde{\Phi}$ is $L_{2,\widetilde{\tau}}$-compact on $L^2(A\rtimes_{\alpha,\mathrm{r}}\Gamma,\widetilde{\tau})$.

"$\Leftarrow$" There is no too many changes in the proof for this direction compared with Proposition 3.4 of [7], because $\widetilde{\Phi}(\lambda_r)=\lambda_r h(r)$, where $\varphi: A\to A$ is not involved. But please pay attention that now we have $\|h(r)\|_{2,\tau}=\|h^*(r)h(r)\|_{2,\tau}^{1/2}=(\tau\circ\mathcal{E}(\Phi^*(\lambda_r)\Phi(\lambda_r)))^{1/2}=\|\widetilde{\Phi}(\lambda_r)\|_{2,\widetilde{\tau}}=\|\widetilde{\Phi}(\lambda_r)-T(\lambda_r)\|_{2,\widetilde{\tau}}\leq\varepsilon$. □

Lemma 2.2. *Suppose that A is a unital C^*-algebra with faithful tracial state τ, $\{\varphi_n : A\to A\}$ is a sequence of maps such that that $\widetilde{\varphi}_n$ converges to $\mathbb{1}$ on $L_2(A,\tau)$ pointwisely, $\{h_n : \Gamma\to\mathcal{Z}(A)\}$ is a sequence of bounded positive definite maps with respect to group action $\alpha:\Gamma\curvearrowright A$. Let $\Phi_n : A\rtimes_{\alpha,r}\Gamma\to A\rtimes_{\alpha,r}\Gamma$ be defined by $\Phi_n(x)=\sum_{t\in\Gamma}\lambda_t\varphi_n(a_t)h_n(t)$, where $x=\sum_{t\in\Gamma}\lambda_t a_t\in C_c(\Gamma,A)$. Then $\{h_n\}$*

converge to the constant function 1 *with respect to* τ *pointwisely on* Γ *if and only if the induced maps* $\{\widetilde{\Phi}_n\}$ *converge to the identity map on* $L^2(A \rtimes_{\alpha,r} \Gamma, \widetilde{\tau})$.

Proof. The proof is essentially the same. I just give the key inequalities.

"$\Rightarrow$" For any $x = \sum_{t\in F} \lambda_t a_t \in C_c(\Gamma, A)$, where $F \subseteq \Gamma$ is finite.

$$\begin{aligned}\|\widetilde{\Phi}_n(x) - x\|_{2,\widetilde{\tau}}^2 =& \|\sum_{t\in F}(\lambda_t\varphi_n(a_t)h_n(t) - \lambda_t a_t)\|_{2,\widetilde{\tau}}^2 \\ =&\tau(\sum_{t\in F}(\varphi_n(a_t)h_n(t) - a_t)^*(\varphi_n(a_t)h_n(t) - a_t)) \\ =&\sum_{t\in F}\|\varphi_n(a_t)h_n(t) - a_t\|_{2,\tau}^2 \\ \leq& 2\sum_{t\in F}(\|\varphi_n(a_t)h_n(t) - \varphi_n(a_t)\|_{2,\tau}^2 + \|\varphi_n(a_t) - a_t\|_{2,\tau}^2)\end{aligned}$$

For any $\varepsilon > 0$, since F is finite, we can find integer $N > 0$ such that $\|h_n(t) - 1\|_{2,\tau} < \varepsilon$, $\|\varphi_n(a_t) - a_t\|_{2,\tau} < \varepsilon$, $\|\varphi_n(a_t)\|_{2,\tau} < M$, for any $t \in F$ and $n > N$. Hence, if $n > N$, we have $\|\widetilde{\Phi}_n(x) - x\|_{2,\widetilde{\tau}}^2 < 2\sum_{t\in F}(M^2\varepsilon^2 + \varepsilon^2)$, i.e. $\|\widetilde{\Phi}_n(x) - x\|_{2,\widetilde{\tau}} < \sqrt{2\sum_{t\in F}(M^2+1)}\varepsilon$.

"$\Leftarrow$" $\|h_n(s) - 1\|_{2,\tau}^2 = \tau((h_n(s) - 1)^*(h_n(s) - 1)) = \|\widetilde{\Phi}_n(\lambda_s) - \lambda_s\|_{2,\widetilde{\tau}}^2 \to 0$. □

□

According to Definition 1.1 and 1.4, it is clear that Theorem 1.5 follows from Lemma 2.1 and Lemma 2.2 above.

3. Concluding Remarks

It is interesting to notice that, since we use a weak version (associated with faithful tracial state) of Haagerup property for C^*-algebra, the corresponding definition of Haagerup property for action turns out to be weak too (both vanishing and converging to 1 tracially). Is it possible to define a strong version of Haagerup property of C^*-algebra that doesn't depend on the choice of faithful tracial states or even is not related to them at all? We know that it is so for the case of von Neumann algebra [4], from which Definition 1.1 is motivated.

Regarding this question, Y. Suzuki [8] recently presented both positive and negative facts about Definition 1.1. On the positive side, he proved that nuclear C^*-algebra would have Haagerup property with respect to any possible faithful tracial state on it; on the negative side, he provided an example of C^*-algebra that has Haagerup property with respect to a faithful tracial state but not so with respect to another one. We have to admit that, without the help of tracial states (especially some canonical ones when dealing with group C^*-algebras and crossed product C^*-algebras), it is rather difficult to measure the approximation of finite-rank maps to the compact one. For example, in Theorem 2.6 of [5], there is an approximation estimation inequality as follows:

For each $x = \sum_{t\in\Gamma} a_t\lambda_\Gamma(t) \in \mathbb{C}[\Gamma] \subseteq C_r^*(\Gamma)$,

$$\|m_{\varphi_i}(x) - m_{\varphi_i^{(k)}}(x)\|_2^2 = \left\| \sum_{t\in\Gamma\setminus F_i^{(k)}} \varphi_i(t)a_t\lambda_\Gamma(t) \right\|_2^2 = \sum_{t\in\Gamma\setminus F_i^{(k)}} |\varphi_i(t)|^2|a_t|^2.$$

If you want to estimate $\sum_{t\in\Gamma\setminus F_i^{(k)}} \varphi_i(t)a_t\lambda_\Gamma(t)$ by the C^*-norm $\|\cdot\|$ in $C^*_{\mathrm{r}}(\Gamma)$, it is rather difficult. So it is not that simple to just require the approximation maps $\{\Phi_i\}$ to be compact on the *operator space* A (actually it is a C^*-algebra) under the C^*-norm. My conjecture is that maybe we can define Haagerup property for C^*-algebras as follows: C^*-algerba A has Haagerup property if there exist c.c.p maps $\{\phi_i : A \to \mathcal{K}\}$ and $\{\psi_i : \mathcal{K} \to A\}$, where $\mathcal{K}$ is the algebra of compact operators, such that $\psi_i \circ \phi_i$ converges to id_A pointwisely. If this works, it is possible to (approximately) factor $A \rtimes_{\alpha,\mathrm{r}} \Gamma$ through $A \otimes \mathcal{K}$, then Dong's approximation technique for Theorem 1.2 may also be applicable to Theorem 1.5. All these would be future considerations.

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